Non-holomorphic multi-matrix gauge invariant operators based on Brauer algebra

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ABSTRACT: We present an orthogonal basis of gauge invariant operators constructed from some complex matrices for the free matrix field, where operators are expressed with the help of Brauer algebra. This is a generalisation of our previous work for a signle complex matrix. We also discuss the matrix quantum mechanics relevant to $\mathcal{N} = 4$ SYM on $S^3 \times R$. A commuting set of conserved operators whose eigenstates are given by the orthogonal basis is shown by using enhanced symmetries at zero coupling.

Contents

1   Introduction

According to the AdS/CFT correspondence, gauge invariant operators correspond to physical states in string theory and, moreover, the spectrum of states is encoded in the anomalous dimension of gauge invariant operators. Of primary interest is to establish this correspondence exactly and to know how the space-time physics can be read from correlators of gauge invariant operators.

     The half-BPS states have attracted much attention because of some observations. The chiral primary sector is described by the complex matrix model which can be obtained by the dimensional reduction from the original $\mathcal{N} = 4$ SYM, which leads to the free fermion

description of this system [4–9]. The chiral primary operators describe KK gravitons and giant gravitons [4, 10–12], whose two-point functions are diagonal [5]. One finds that a particular linear combination of single traces and multi-traces corresponds a single giant graviton or a set of giant gravitons, whose information is offered by Young tableau [5]. A Young diagram plays the role of organising the multi-trace structure of this sector and completely determines the space-time physics.

A next question along this line is how we can expand the chiral primary sector to more general classes of gauge invariant operators. It is interesting to ask how group theory can be utilised to organise gauge invariant operators to make the connection to string theory more manifest. In [1], the chiral primary operator was extended to include the complex conjugation of a complex matrix, and an orthogonal complete set was constructed using the free field correlator. Because replacing X by $X^\dagger$ in chiral primary operators means to change the sign of angular momentum and the coupling to the background field in the dual string side, this sector would describe physics containing giant gravitons and anti-giant gravitons. If the interactions are taken into account properly, one may expect to see the instability originated from tachyons appearing between those branes.

A mathematical new element in [1] was to use Brauer algebra to organise the multi-trace structure of gauge invariant operators constructed from X and $X^\dagger$. This is a generalisation of the fact that the symmetric group played a role to organise the multi-trace structure of the chiral primary sector. The Brauer algebra contains the contraction in addition to the group algebra of the symmetric group, which manages gauge invariant operators involving X and $X^\dagger$ in a trace. The complete set constructed in [1] has more group theoretic labels than the chiral primary sector, which is reflected in the fact that the operators were conjectured to describe a system of branes and anti-branes. Other studies of diagonalising the two-point functions in the free field limit have been reported in [13–17].

In this paper, we continue this line to explore non-holomorphic gauge invariant operators built from some complex matrices. We work out the construction of diagonal two-point functions at the free level of the matrix field theory and see what kind of group theoretic structure show up.

Here are the outline of this paper. The construction of the orthogonal set of gauge invariant operators is reviewed in section 2 for the chiral primary sector and the non-holomorphic extension. In section 3, we shall give the construction of an orthogonal set for the non-holomorphic multi-matrix sector. Section 4 is given to show some examples of our operator. The simplest sector contains the Konishi operator. In section 5, the multi-matrix quantum mechanics is discussed. In our previous paper [3], we have introduced the concept of enhanced symmetries at the free level and have given some operators commuting with the Hamiltonian, which were exploited to measure the spectrum of orthogonal bases. It will be shown how the orthogonal gauge invariant operators constructed in this paper are characterised by conserved charges based on the enhanced symmetries. In section 6, we discuss some future problems. Some detailed calculations are shown in appendices.

2 Diagonal basis of gauge invariant operators built from X and $X^\dagger$ based on Brauer algebra: review

In this section, we review the chiral primary sector and the Brauer basis for X and $X^\dagger$ sector.

Operators corresponding to the highest weight in the half-BPS sector are given by holomorphic gauge invariant operators made from a complex matrix X. The diagonal basis of operators constructed from n matrices is labelled by a Young diagram R with n boxes [5]. We shall review the construction of this basis from a group theoretical point of view.

An $N \times N$ matrix X can be viewed as an endomorphism acting on an N-dimensional vector space V, i.e. $X: V \to V$. The tensor product $X^{\otimes n}$ acts on $V^{\otimes n}$. It is useful to introduce the symmetric group S_n as a tool to organise both single trace and multi-trace. We let elements σ of the symmetric group S_n act on $V^{\otimes n}$ as the permutations of n vector spaces V. One easily finds that any gauge invariant operator constructed from n X's can be expressed as $tr_n(\sigma X^{\otimes n}) = X^{i_1}_{i_{\sigma(1)}} \cdots X^{i_n}_{i_{\sigma(n)}}$ where the trace tr_n is taken in $V^{\otimes n}$. It is noted that $tr_n(h\sigma h^{-1} X^{\otimes n})$ also gives the same gauge invariant operator for any $h \in S_n$, which means the conjugacy classes of S_n classify gauge invariant operators.

It is convenient to start with the following Schur-Weyl duality to introduce the representation basis,

$$V^{\otimes n} = \bigoplus_R V_R^{\mathrm{U}(N)} \otimes V_R^{S_n}. \tag{2.1}$$

This comes from the fact that the symmetric group S_n is the centraliser of $\mathrm{U}(N)$ on $V^{\otimes n}$. R runs over all irreducible representations with n boxes satisfying $c_1(R) \le N$, where $c_1(R)$ is the length of the first column of R. The projection operator associated with an irreducible representation R can be expressed as an element in the group algebra of the symmetric group S_n as

$$p_R = \frac{d_R}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma)\sigma. \tag{2.2}$$

Consider the following particular linear combination of $tr_n(\sigma X^{\otimes n})$

$$O_R(X) := tr_n(p_R X^{\otimes n}). \tag{2.3}$$

It was shown in [5] that it has the diagonal two-point function:

$$\langle O_R(X)^\dagger O_S(X)\rangle = \delta_{\mathrm{RS}} n_R! d_R Dim R, \tag{2.4}$$

which can be shown using the propagator of X_{ij}:[1]

$$\langle X_{ij}^\dagger X_{kl}\rangle = \delta_{jk}\delta_{il}. \tag{2.5}$$

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[1]In this paper, we study the theory with gauge group $\mathrm{U}(N)$. See [19, 20] for the extension to gauge group $\mathrm{SU}(N)$.

The space-time dependence has been ignored because it is easily recovered from the conformal invariance, and we are mainly interested in the colour and flavour dependence. Some properties of the projector are very helpful to calculate correlation functions [18].

We extend this sector to include $X^\dagger$. The construction of an orthogonal set for non-holomorphic operators in the free matrix field theory was completed in [1] by introducing the Brauer algebra. The Brauer algebra is an algebra which has the group algebra of $S_m \times S_n$ and contractions.[2] It is reviewed in section 3 of [1].

We start with the fact that any multi-trace gauge invariant operator constructed from m copies of X and n copies of $X^\dagger$ can be indicated through an element of the Brauer algebra as $tr_{m,n}(bX^{\otimes m} \otimes X^{*\otimes n})$, where $tr_{m,n}$ is taken in the space of $V^{\otimes m} \otimes \bar{V}^{\otimes n}$. At $m = n = 1$, b takes 1 (the unit element) and C (contraction), which give $trX trX^\dagger$ and $tr(XX^\dagger)$, respectively. We note that $tr_{m,n}(h^{-1}bhX^{\otimes m} \otimes X^{*\otimes n})$ for any $h \in S_m \times S_n$ provides the same gauge invariant operator.[3] This means multi-trace operators are classified by the equivalence classes under the conjugation of the symmetric group $S_m \times S_n$.

The Brauer algebra[4] can be introduced as the centraliser of $\mathrm{U}(N)$ acting on $V^{\otimes m} \otimes \bar{V}^{\otimes n}$,

$$V^{\otimes m} \otimes \bar{V}^{\otimes n} = \bigoplus_\gamma V_\gamma^{\mathrm{U}(N)} \otimes V_\gamma^{B_N(m,n)}. \tag{2.6}$$

The sum is over irreducible representations γ of $\mathrm{U}(N)$ and $B_N(m,n)$. This equation follows from the fact that Brauer elements commute with the action of $\mathrm{U}(N)$ on $V^{\otimes m} \otimes \bar{V}^{\otimes n}$. The irreducible representation γ is determined by a set (γ_+, γ_-, k), where γ_+ is a partition of $m - k$, γ_- is a partition of $n - k$ and k is an integer with $0 \le k \le \min(m, n)$. These definitions provide a constraint $c_1(\gamma_+) + c_1(\gamma_-) \le N$. Because the Wick contractions are symbolised in terms of the symmetric group $S_m \times S_n$ (see, e.g. (3.8)), it will be convenient to decompose the Brauer algebra into the group algebra of $S_m \times S_n$ (which we shall denoted by $\mathbb{C}(S_m \times S_n)$) as

$$V_\gamma^{B_N(m,n)} = \bigoplus_A V_A^{\mathbb{C}(S_m \times S_n)} \otimes V_{\gamma \to A}. \tag{2.7}$$

The sum is taken over irreducible representations A of the symmetric group, and $V_{\gamma \to A}$ represents the space of the multiplicity associated with the decomposition. We shall express an irreducible representation A of $S_m \times S_n$ as a set of a partition of m and a partition of n: (α, β). The multiplicity of the irreducible representation $A = (\alpha, \beta)$ of $\mathbb{C}[S_m \times S_n]$ appearing in the irreducible representation γ of $B_N(m, n)$ is read from the formula

$$M_A^\gamma := Dim(V_{\gamma \to A}) = \sum_{\delta \vdash k} g(\delta, \gamma_+; \alpha) g(\delta, \gamma_-; \beta). \tag{2.8}$$

Here $\delta \vdash k$ is shorthand to express that δ is a partition of k. The Littlewood-Richardson coefficient $g(\delta, \gamma_+; \alpha)$ represents the multiplicity of the representation α appearing in the

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[2]The contraction C is a linear map from $V \otimes \bar{V}$ to itself. The action is $Cv_i \otimes \bar{v}_j = \delta_{ij} v_k \otimes \bar{v}_k$.

[3]Using the cyclicity property of the trace, the conjugate action of h on b results in the conjugate action on $X^{\otimes m} \otimes X^{*\otimes n}$, which is re-ordering of X's and/or that of X^*'s.

[4]The Brauer algebra is sensitive to N. For example, we have $C^2 = NC$ for a contraction C.

tensor product of the representations δ and γ_+. This formula states $\gamma_+ = \alpha$ and $\gamma_- = \beta$ at $k = 0$.

An orthogonal and complete set introduced in [1] is

$$O^\gamma_{A,ij}(X, X^*) =: tr_{m,n}(Q^\gamma_{A,ij} X^{\otimes m} \otimes X^{*\otimes n}) :, \tag{2.9}$$

where $:$ means that divergences (self-contractions) associated with defining a composite operator are omitted, i.e. $\langle : tr(\ ) : \rangle = 0$. The indices i, j run over 1 to M^γ_A, behaving like matrix indices as $Q^\gamma_{A,ij} Q^{\gamma'}_{A',kl} = \delta_{\gamma\gamma'} \delta_{AA'} \delta_{jk} Q^\gamma_{A,il}$. When the multiplicity is trivial, this operator becomes a projector, which happens at $k = 0$, $k = m = n$, and so on.

We introduce the restricted character $\chi^\gamma_{A,ij}(b)$ which is defined by

$$\chi^\gamma_{A,ij}(b) = \sum_{m_A} \langle \gamma \to A, m_A, i|b|\gamma \to A, m_A, j\rangle, \tag{2.10}$$

where $|\gamma \to A, m_A, j\rangle$ is a state of the Brauer algebra in the irreducible representation γ associated with the decomposition to the subalgebra $\mathbb{C}(S_m \times S_n)$. m_A represents components in A. $\chi_\gamma(b) = \sum_{A,i} \chi^\gamma_{A,ii}(b)$ is the character of the Brauer algebra. The restricted character enables us to express $Q^\gamma_{A,ij}$ in terms of elements in the Brauer algebra as[5]

$$Q^\gamma_{A,ij} = t_\gamma \sum_b \chi^\gamma_{A,ji}(b) b^*. \tag{2.11}$$

t_γ is the dimension of the U(N) irreducible representation γ. b^* is the dual element of b which is specified by $tr_{m,n}(bb^*) = 1$. For more information about this dual element, see [1, 42]. The operator $Q^\gamma_{A,ij}$ commutes with any element of $S_m \times S_n$:

$$hQ^\gamma_{A,ij} = Q^\gamma_{A,ij} h, \quad h \in \mathbb{C}(S_m \times S_n), \tag{2.12}$$

which can be shown by exploiting $hb^* h^{-1} = (hbh^{-1})^*$. The property (2.12) is significant to show the diagonal two-point function. A projector associated with an irreducible representation γ can be written down as

$$P^\gamma = t_\gamma \sum_b \chi_\gamma(b) b^* = \sum_{A,i} Q^\gamma_{A,ii}. \tag{2.13}$$

Some remarks are in order. The free two-point functions are shown to be diagonal:

$$\langle O^\gamma_{A,ij}(X, X^*)^\dagger O^{\gamma'}_{A',i'j'}(X, X^*)\rangle = m!n! d_A t_\gamma \delta_{\gamma\gamma'} \delta_{AA'} \delta_{ii'} \delta_{jj'}, \tag{2.14}$$

where d_A is the dimension of $S_m \times S_n$ associated with the irreducible representation A.

There is a special sub-class of the operator $Q^\gamma_{A,ij}$, which is parametrised by $k = 0$. In this case, $\gamma = A$, i.e. $\gamma_+ = \alpha$, $\gamma_- = \beta$ (see the comment below (2.8)). It is therefore labelled by two Young diagrams. We shall denote it by $P_{\alpha\beta}$. Because the leading term of $P_{\alpha\beta}$ is found to be $p_\alpha p_\beta$, this gives the product of the holomorphic operator and the anti-holomorphic operator $O_\alpha(X) O_\beta(X^\dagger)$. The $k = 0$ projector $P_{\alpha\beta}$ also plays an important role

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[5]A typo in (122) of [3] is corrected here.

in the context of the large N expansion of two-dimensional Yang-Mills [21]. Using some properties of the Brauer algebra, a new expression of the SU(N) dimension was given, leading to a new formulation of two-dimensional Yang-Mills [2].

This class of gauge invariant operators does not require the normal ordering prescription to make it well-defined as composite operators. In other words we can show

$$: tr_{m,n}(P_{\alpha\beta}X^{\otimes m} \otimes X^{*\otimes n}) := tr_{m,n}(P_{\alpha\beta}X^{\otimes m} \otimes X^{*\otimes n}). \tag{2.15}$$

Considering the fact that the Wick contraction between an X and an X^* is performed by a contraction C, $CP_{\alpha\beta} = 0$ is a sufficient condition for the above equation. It indeed follows from the fact that the projector $P_{\alpha\beta}$ is orthogonal to the other projectors relevant to $k \neq 0$.

We conclude this section by showing the simplest example of $m = n = 1$:

$$trX trX^\dagger - \frac{1}{N}tr(XX^\dagger), \quad \frac{1}{N}tr(XX^\dagger). \tag{2.16}$$

The first one is labelled by $k = 0$ and $A = ([1],[1])$, while the second one is by $\gamma = (\emptyset, \emptyset, 1)$ and $A = ([1],[1])$.

3 Diagonal basis of non-holomorphic multi-matrix gauge invariant operators

In the previous section, we have reviewed a specific set of gauge invariant operators made from X and $X^\dagger$ that are engineered by the Brauer algebra. In this section we generalise this to multi-matrix models. We shall present an orthogonal set of operators composed of X_a and $X_a^\dagger$, where $a = 1, \ldots, p$, at the free coupling. This has an additional flavour index compared to the previous case. Hence we first work out the flavour structure, and later we will move to the colour structure. This procedure exploits the approach of [13] to deal with global indices. When p is 3, it is relevant to the SO(6) sector of $\mathcal{N} = 4$ four-dimensional super Yang-Mills theory.

3.1 Gauge covariant operators in representation basis

We start with the following gauge covariant operator

$$(O_{\vec{a},\vec{b}})^I_J := (X_{a_1})^{i_1}_{j_1} \otimes \cdots \otimes (X_{a_m})^{i_m}_{j_m} \otimes (X^*_{b_1})^{i_{m+1}}_{j_{m+1}} \otimes \cdots \otimes (X^*_{b_n})^{i_{m+n}}_{j_{m+n}}, \tag{3.1}$$

where $a_i, b_i = 1, 2, \ldots, p$. We will rewrite the flavour structure using a representation basis.

Let V_F be the space of the fundamental representation of U(p). The following Schur-Weyl duality is relevant to the flavour structure:

$$V_F^{\otimes m} \otimes \bar{V}_F^{\otimes n} = \bigoplus_\Lambda V_\Lambda^{U(p)} \otimes V_\Lambda^{B_p(m,n)}. \tag{3.2}$$

Here the sum runs over irreducible representations of U(p) and $B_p(m,n)$. The representation Λ is labelled by a set $(\Lambda_+, \Lambda_-, l)$, where l is an integer with $0 \leq l \leq \min(m,n)$, and Λ_+ and Λ_- are given by a partition of $m - l$ and a partition of $n - l$. It is noted that this

Brauer algebra $B_p(m,n)$ should not be confused with another Brauer algebra $B_N(m,n)$ which is relevant for the colour structure.

The group algebra of $S_m \times S_n$, $\mathbb{C}(S_m \times S_n)$, is a subalgebra of $B_p(m,n)$, hence we consider the decomposition of $B_p(m,n)$ into $\mathbb{C}(S_m \times S_n)$:

$$V_\Lambda^{B_p(m,n)} = \bigoplus_{\Lambda_1} V_{\Lambda_1}^{\mathbb{C}(S_m \times S_n)} \otimes V_{\Lambda \to \Lambda_1}, \tag{3.3}$$

where Λ_1 runs over irreducible representations of $\mathbb{C}(S_m \times S_n)$. The second factor in the right-hand side represents the space of the multiplicity arising from this decomposition. Combining (3.2) and (3.3), we have the following equation for the flavour structure

$$V_F^{\otimes m} \otimes \bar{V}_F^{\otimes n} = \bigoplus_{\Lambda,\Lambda_1} V_\Lambda^{U(p)} \otimes V_{\Lambda_1}^{\mathbb{C}(S_m \times S_n)} \otimes V_{\Lambda \to \Lambda_1}. \tag{3.4}$$

Based on this decomposition, one may introduce a covariant operator in a representation basis as

$$O_{\vec{a},\vec{b}} = \sum_{\Lambda,M_\Lambda,\Lambda_1,m_{\Lambda_1},\tau} C_{\vec{a},\vec{b}}^{\Lambda,M_\Lambda,\Lambda_1,m_{\Lambda_1},\tau} O_{\Lambda,M_\Lambda,\Lambda_1,m_{\Lambda_1},\tau}. \tag{3.5}$$

M_Λ represents states in the irreducible representation Λ of $U(p)$, and m_{Λ_1} runs over states in the Λ_1. τ is an index running over the multiplicity of Λ_1 in Λ. The inverse is

$$O_{\Lambda,M_\Lambda,\Lambda_1,m_{\Lambda_1},\tau} = \sum_{\vec{a},\vec{b}} C_{\Lambda,M_\Lambda,\Lambda_1,m_{\Lambda_1},\tau}^{\vec{a},\vec{b}} O_{\vec{a},\vec{b}}. \tag{3.6}$$

We now calculate the free two-point function of the operator. Using

$$\langle (X_a^\dagger)_{ij} (X_b)_{kl} \rangle = \delta_{ab} \delta_{jk} \delta_{il}, \tag{3.7}$$

we get for $O_{\vec{a},\vec{b}}$

$$\langle : (O_{\vec{a},\vec{b}}^\dagger)_J^I :: (O_{\vec{a}',\vec{b}'})_L^K : \rangle = \sum_{\sigma \in S_m \times S_n} \prod_{k=1}^m \delta_{a_k a'_{\sigma(k)}} \prod_{l=1}^n \delta_{b_l b'_{\sigma(l)}} (\sigma)_J^K (\sigma^{-1})_L^I, \tag{3.8}$$

where

$$(\sigma)_J^K = (\sigma)_{j_1 \cdots j_{m+n}}^{k_1 \cdots k_{m+n}} := \delta_{j_{\sigma(1)}}^{k_1} \cdots \delta_{j_{\sigma(m+n)}}^{k_{m+n}}. \tag{3.9}$$

The two-point function of the representation basis (3.6) can be computed as

$$\langle : (O_{\Lambda,M_\Lambda,\Lambda_1,m_{\Lambda_1},\tau}^\dagger)_J^I :: (O_{\Lambda',M'_{\Lambda'},\Lambda'_1,m'_{\Lambda'_1},\tau'})_L^K : \rangle$$
$$= \delta_{\Lambda\Lambda'} \delta_{M_\Lambda M'_{\Lambda'}} \delta_{\Lambda_1 \Lambda'_1} \delta_{\tau\tau'} \sum_{\sigma \in S_m \times S_n} D_{m_{\Lambda_1} m'_{\Lambda'_1}}^{\Lambda_1}(\sigma)(\sigma)_J^K (\sigma^{-1})_L^I. \tag{3.10}$$

This will be proved in appendix B.1.

3.2 Gauge invariant operators in representation basis

All gauge invariant operators constructed from the covariant operators are expressed by $tr_{m,n}(bO_{\Lambda,M_\Lambda,\Lambda_1,m_{\Lambda_1},\tau})$. To rewrite the colour structure in terms of a representation basis, we again use the decomposition given in (2.6) and (2.7):

$$V^{\otimes m} \otimes \bar{V}^{\otimes n} = \bigoplus_{\gamma,A} V_\gamma^{U(N)} \otimes V_A^{\mathbb{C}(S_m \times S_n)} \otimes V_{\gamma \to A}. \tag{3.11}$$

We thus have two kinds of irreducible representations of the symmetric group. One, Λ, is responsible for the global indices, and the other, A, is for the colour indices. These two kinds of representations come from two different actions of the symmetric group which are related each other as we shall discuss later (see (3.18) and appendix C). We now consider the inner tensor product of $A \otimes A$ decomposing into Λ_1 and denote the multiplicity of Λ_1 by τ_{Λ_1}. Introducing the Clebsch-Gordan coefficient[6] $C_{A,m_A,A,m'_A}^{\tau_{\Lambda_1},\Lambda_1,m_{\Lambda_1}}$ associated with this decomposition, we propose the following gauge invariant operator

$$\mathcal{O}_{\Lambda,M_\Lambda,\Lambda_1,\tau}^{\gamma,A,ij,\tau_{\Lambda_1}} = \sum_{m_A,m'_A,m_{\Lambda_1}} C_{A,m_A,A,m'_A}^{\tau_{\Lambda_1},\Lambda_1,m_{\Lambda_1}} \sum_{b \in B_N(m,n)} D_{A,m_A,j,m'_A,i}^{\gamma}(b^*) : tr_{m,n}(bO_{\Lambda,M_\Lambda,\Lambda_1,m_{\Lambda_1},\tau}) : . \tag{3.12}$$

Here

$$D_{A,m_A,j,m'_A,i}^{\gamma}(b) := \langle \gamma \to A, m_A, j | b | \gamma \to A, m'_A, i \rangle. \tag{3.13}$$

A remark is

$$\chi_{A,ji}^{\gamma}(b) = \sum_{m_A} D_{A,m_A,j,m_A,i}^{\gamma}(b) \tag{3.14}$$

is the restricted character in (2.10).

We can show that the two-point function of the operator (3.12) is diagonal:

$$\langle \mathcal{O}_{\Lambda,M_\Lambda,\Lambda_1,\tau}^{\gamma,A,ij,\tau_{\Lambda_1}\,\dagger} \mathcal{O}_{\Lambda',M_{\Lambda'},\Lambda'_1,\tau'}^{\gamma',A',i'j',\tau'_{\Lambda'_1}} \rangle = m!n!d_{\Lambda_1}\frac{1}{t^\gamma}\delta_{\gamma\gamma'}\delta_{AA'}\delta_{\Lambda\Lambda'}\delta_{M_\Lambda M'_{\Lambda'}}\delta_{\Lambda_1\Lambda'_1}\delta_{\tau_{\Lambda_1}\tau'_{\Lambda'_1}}\delta_{\tau\tau'}\delta_{ii'}\delta_{jj'}. \tag{3.15}$$

This is the main result of this paper. We shall give the proof to the appendix B.

For $p = 1$, the operator should coincide to the operator seen in the previous section. In this case, Λ is a one-dimensional representation, so the covariant operator is labelled by Λ alone i.e. the number of X and that of X^*. The Clebsh-Gordan for this case is given by

$$C_{A,m_A,A,m'_A} = \frac{1}{\sqrt{d_A}}\delta_{m_A m'_A}. \tag{3.16}$$

We thus obtain

$$\mathcal{O}_{\gamma,A,ij} = \frac{1}{\sqrt{d_A}}\sum_b \chi_{A,ji}^{\gamma}(b^*)tr_{m,n}(bX^{\otimes m} \otimes X^{*\otimes n}), \tag{3.17}$$

which is equivalent to (2.9) up to the normalisation factor.

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[6]See [22] for some properties of the Clebsch-Gordan coefficient.

Before proceeding to the next subsection, we summarise the relation between the two actions of the symmetric group. Two kinds of Brauer algebras have been introduced, one is relevant for the flavour structure, and the other is for the colour structure. Both Brauer algebras contain the same sub-algebra, that is, the group algebra of $S_m \times S_n$, but they admit different actions on the operator. The permutation acting on the flavour indices is equivalent to re-ordering of X's and/or that of X^*'s. If we act with permutations on upper colour indices and lower colour indices simultaneously, we effectively get permutations on flavour indices:

$$\left(O_{\vec{a},\vec{b}}\right)^{\sigma(I)}_{\sigma(J)} = \left(O_{\sigma(\vec{a}),\sigma(\vec{b})}\right)^{I}_{J}. \tag{3.18}$$

In appendix C we shall confirm that the gauge invariant operator respects this symmetry.

3.3   Operator in the $k = 0$ representation of γ

Let us now study the special class of the operator where the representation of γ is specified by $k = 0$. The multiplicity indices related to the decomposition $\gamma \to A$ can go away because $\gamma = A$ from the formula (2.8) at $k = 0$. So the operator is simplified to be

$$
\begin{aligned}
\mathcal{O}^{A,\tau_{\Lambda_1}}_{\Lambda,M_\Lambda,\Lambda_1,\tau} &= C^{\tau_{\Lambda_1},\Lambda_1,m_{\Lambda_1}}_{A,m_A,A,m'_A} \sum_{b \in B_N(m,n)} D^A_{m_A,m'_A}(b) tr_{m,n}(b^* O_{\Lambda,M_\Lambda,\Lambda_1,m_{\Lambda_1},\tau}) \\
&= C^{\tau_{\Lambda_1},\Lambda_1,m_{\Lambda_1}}_{A,m_A,A,m'_A} \sum_{\alpha \in S_m \times S_n} D^A_{m_A,m'_A}(\alpha) tr_{m,n}(\alpha^* O_{\Lambda,M_\Lambda,\Lambda_1,m_{\Lambda_1},\tau}) \\
&= C^{\tau_{\Lambda_1},\Lambda_1,m_{\Lambda_1}}_{A,m_A,A,m'_A} \sum_{\alpha \in S_m \times S_n} D^A_{m_A,m'_A}(\alpha) tr_{m,n}(1^*\alpha^{-1} O_{\Lambda,M_\Lambda,\Lambda_1,m_{\Lambda_1},\tau}). \tag{3.19}
\end{aligned}
$$

$D^A_{m_A,m'_A}(C) = 0$ has been used to get the second equality. The third equality follows from the formula $\alpha^* = (1^*)\alpha^{-1}$ for $\alpha \in S_m \times S_n$ which is derived in [1].

For the complete set of the X and $X^\dagger$ sector, we could show that operators corresponding to the $k = 0$ representation do not require the normal ordering prescription because short distance singularities arising from self-contractions vanish. This property also holds in this case. Using the fact that the Wick contraction between an X and an X^* can be expressed by a contraction C, we show that replacing a set of an X and an X^* by a contraction C in (3.19) vanishes:

$$
\begin{aligned}
&\sum_\alpha D^A_{m_A,m'_A}(\alpha) tr_{m,n}(C1^*\alpha^{-1} O') \\
&= \sum_\alpha D^A_{m_A,m'_A}(\alpha) \sum_{\gamma'} \chi^{\gamma'}(C1^*\alpha^{-1}) tr_{m,n}(P^{\gamma'} O') \\
&= \sum_\alpha \sum_{\gamma'} D^A_{m_A,m'_A}(\alpha) D^{\gamma'}_{IJ}(C1^*) D^{\gamma'}_{JI}(\alpha^{-1}) tr_{m,n}(P^{\gamma'} O') \\
&\propto D^A_{m_A,m'_A}(C1^*) = 0. \tag{3.20}
\end{aligned}
$$

Here the first equality comes from the decomposition (2.6), and we have used the orthogonality of the representation matrix (B.14) in the last step. In this way, we can demonstrate that the normal ordering prescription is no longer needed for gauge invariant operators belonging to the $k = 0$ representation.

Another property of the $k = 0$ sector is that the leading term contains the product of a holomorphic operator and an anti-holomorphic operator. We shall show that the operator constructed in [13] will be reproduced as a subset of our operator. To show this, we use an explicit expression of 1^*:

$$1^* = \frac{1}{N^{m+n}} \Omega_m^{-1} \Omega_n^{-1} + \cdots, \tag{3.21}$$

where Ω_m is a central element in the group algebra of S_m, and $\cdots$ are terms which are not in $\mathbb{C}(S_m \times S_n)$ but in $B_N(m, n)$. See appendix A in [1] about more concrete form of 1^* for some examples. We substitute the leading term of 1^* into (3.19) and restrict to the $l = 0$ sector of the representation Λ. This implies we do not see terms in which flavour indices are contracted between X and $X^\dagger$. This is the case of $\Lambda = \Lambda_1$, so the index τ can be suppressed.

The operator (3.19) with the above restrictions taken into account becomes

$$\frac{1}{N^{m+n}} C_{A,m_A,A,m_A'}^{\tau_{\Lambda_1}, \Lambda_1, m_{\Lambda_1}} \sum_{\alpha \in S_m \times S_n} D_{m_A, m_A'}^A(\alpha) tr_{m,n}(\Omega_m^{-1} \Omega_n^{-1} \alpha^{-1} O_{\Lambda_1, m_{\Lambda_1}})$$

$$= \frac{1}{N^{m+n}} C_{A,m_A,A,m_A'}^{\tau_{\Lambda_1}, \Lambda_1, m_{\Lambda_1}} \sum_{\alpha \in S_m \times S_n} D_{m_A, m_A'}^A(\Omega_m^{-1} \Omega_n^{-1} \alpha) tr_{m,n}(\alpha^{-1} O_{\Lambda_1, m_{\Lambda_1}})$$

$$= \frac{1}{N^{m+n}} C_{A,m_A,A,m_A'}^{\tau_{\Lambda_1}, \Lambda_1, m_{\Lambda_1}} \frac{1}{d_A} \chi_A\left(\Omega_m^{-1} \Omega_n^{-1}\right) \sum_{\alpha \in S_m \times S_n} D_{m_A, m_A'}^A(\alpha) tr_{m,n}(\alpha^{-1} O_{\Lambda_1, m_{\Lambda_1}})$$

$$= \frac{1}{N^{m+n}} C_{A,m_A,A,m_A'}^{\tau_{\Lambda_1}, \Lambda_1, m_{\Lambda_1}} \frac{1}{d_A} \chi_A\left(\Omega_m^{-1} \Omega_n^{-1}\right) \sum_{\sigma \in S_m, \tau \in \times S_n} D_{m_A, m_A'}^A(\sigma \otimes \tau) tr_{m,n}(\sigma^{-1} \otimes \tau^{-1} O_{\Lambda_1, m_{\Lambda_1}}).$$

Each factor can be factorised into the S_m part and the S_n part. For example, the character can be rewritten as

$$\chi_A\left(\Omega_m^{-1} \Omega_n^{-1}\right) = \chi_R\left(\Omega_m^{-1}\right) \chi_S(\Omega_n^{-1})$$

$$= \frac{1}{m! n!} \frac{N^{m+n} d_R^2 d_S^2}{DimR DimS}, \tag{3.22}$$

where we have expressed A as (R, S) where R is an irreducible representation of S_m and S is an irreducible representation of S_n, and the formula $DimR = \frac{N^m}{m!} \chi_R(\Omega_m)$ has been used. Taking this factorisation into account, we find that the operator in the $k = 0$ representation of γ and the $l = 0$ representation of Λ contains the product of the holomorphic multi-matrix operator and the anti-holomorphic multi-matirx operator presented in [13, 17].

4 Examples

In this section, we present explicit forms of our operator for two simple cases. We will set $p = 3$ to see a connection to the $\mathcal{N} = 4$ SYM.

4.1 $m = 1$, $n = 1$

In this case, there are only two cases for $\Lambda = (\Lambda_+, \Lambda_-, l)$, i.e. $([1], [1], 0)$ and $(\emptyset, \emptyset, 1)$. We call them $l = 0$ and $l = 1$ because the integer l completely identifies them.

The bases $|\Lambda, M_\Lambda\rangle$ are

$$|l = 0\rangle = |a, b\rangle - \frac{1}{3}\delta_{ab}\sum_{c=1}^{3}|c, c\rangle,$$

$$|l = 1\rangle = \frac{1}{3}\delta_{ab}\sum_{c=1}^{3}|c, c\rangle. \qquad (4.1)$$

Using the above states, the Clebsch-Gordan coefficient $C_{\Lambda,M_\Lambda}^{a',b'} = \langle a', b'|\Lambda, M_\Lambda\rangle$ can be calculated as

$$C_{l=0}^{a',b'} = \langle a', b'|l = 0\rangle = \delta_{aa'}\delta_{bb'} - \frac{1}{3}\delta_{ab}\delta_{a'b'},$$

$$C_{l=1}^{a',b'} = \langle a', b'|l = 1\rangle = \frac{1}{3}\delta_{ab}\delta_{a'b'}, \qquad (4.2)$$

which yield covariant operators $O_{\Lambda,M_\Lambda} = C_{\Lambda,M_\Lambda}^{a',b'}(X_{a'} \otimes X_{b'})$:

$$O_{l=0} = X_a \otimes X_b^* - \frac{1}{3}\delta_{ab}\sum_{c'}X_{c'} \otimes X_{c'}^*,$$

$$O_{l=1} = \frac{1}{3}\delta_{ab}\sum_{c'}X_{c'} \otimes X_{c'}^*. \qquad (4.3)$$

We next work out the gauge invariant operator. For this case the operator (3.12) is simplified to

$$\mathcal{O}_{\Lambda,M_\Lambda}^{\gamma} = \sum_{b \in B_N(1,1)} D^\gamma(b^*)tr_{1,1}(bO_{\Lambda,M_\Lambda}) = \frac{1}{t_\gamma}tr_{1,1}(P^\gamma O_{\Lambda,M_\Lambda}). \qquad (4.4)$$

γ is specified by an integer k, which takes either $k = 0$ or $k = 1$. P^γ is a projector associated with γ:

$$P^{(k=0)} = 1 - \frac{C}{N}, \quad P^{(k=1)} = \frac{C}{N}. \qquad (4.5)$$

Re-normalising the operators as $\mathcal{O}^\gamma \to t_\gamma \mathcal{O}^\gamma$ for convenience, we reach the following gauge invariant operators

$$\mathcal{O}_{k=0,l=0} = (trX_a)(trX_b^\dagger) - \frac{1}{N}tr(X_aX_b^\dagger) - \frac{1}{3}\delta_{ab}(trX_c)(trX_c^\dagger) + \frac{1}{3N}\delta_{ab}tr(X_cX_c^\dagger)$$

$$\mathcal{O}_{k=1,l=0} = \frac{1}{N}\left(tr(X_aX_b^\dagger) - \frac{1}{3}\delta_{ab}tr(X_cX_c^\dagger)\right)$$

$$\mathcal{O}_{k=0,l=1} = \frac{1}{3}\delta_{ab}\left((trX_c)(trX_c^\dagger) - \frac{1}{N}tr(X_cX_c^\dagger)\right)$$

$$\mathcal{O}_{k=1,l=1} = \frac{1}{3N}\delta_{ab}tr(X_cX_c^\dagger). \qquad (4.6)$$

In fact, these are not eigenstates of the one-loop dilatation operator.[7] Correct eigenstates at one-loop are

$$\mathcal{O}_{k=0,l=0}$$

---

[7]The one-loop dilatation operator in the so(6) sector will be rewritten in terms of the complex variables in appendix A.

$$\mathcal{O}_{k=1,l=0}$$

$$\mathcal{O}_{k=0,l=1} + \mathcal{O}_{k=1,l=1} = \frac{1}{3}\delta_{ab}(trX_c)(trX_c^\dagger)$$

$$\frac{N^2-1}{N}\mathcal{O}_{k=1,l=1} - \frac{1}{N}\mathcal{O}_{k=0,l=1} = \frac{1}{3}\delta_{ab}\left(tr(X_cX_c^\dagger) - \frac{1}{N}(trX_c)(trX_c^\dagger)\right) \qquad (4.7)$$

with eigenvalues 0, 0, 0, $6/\sqrt{3}$, respectively. The first three are operators in the short multiplets. The last one is the Konishi operator. The $l = 0$ representations are not mixed up with the $l = 1$ representations because the Hamiltonian commutes with the Brauer algebra governing the flavour structure.

4.2 $m = 2$, $n = 1$

We next show the case at $m = 2$, $n = 1$. States $|\Lambda, \Lambda_1\rangle$ are

$$|[2],[1],0\rangle = \frac{1}{2}(|a,b,c\rangle + |b,a,c\rangle) - \frac{1}{8}\delta_{ac}\sum_{d=1}^{3}(|d,b,d\rangle + |b,d,d\rangle)$$

$$-\frac{1}{8}\delta_{bc}\sum_{d=1}^{3}(|a,d,d\rangle + |d,a,d\rangle)$$

$$|[1,1],[1],0\rangle = \frac{1}{2}(|a,b,c\rangle - |b,a,c\rangle) - \frac{1}{4}\delta_{ac}\sum_{d=1}^{3}(|d,b,d\rangle - |b,d,d\rangle)$$

$$-\frac{1}{4}\delta_{bc}\sum_{d=1}^{3}(|a,d,d\rangle - |d,a,d\rangle)$$

$$|([1],\emptyset,1),([2],[1])\rangle = \frac{1}{8}\delta_{ac}\sum_{d=1}^{3}(|d,b,d\rangle + |b,d,d\rangle) + \frac{1}{8}\delta_{bc}\sum_{d=1}^{3}(|a,d,d\rangle + |d,a,d\rangle)$$

$$|([1],\emptyset,1),([1,1],[1])\rangle = \frac{1}{4}\delta_{ac}\sum_{d=1}^{3}(|d,b,d\rangle - |b,d,d\rangle) + \frac{1}{4}\delta_{bc}\sum_{d=1}^{3}(|a,d,d\rangle - |d,a,d\rangle).$$

Projectors in (4.11) and (4.13) with N replaced by $3(=p)$ are useful to calculate the above equations. Recall that $\Lambda = \Lambda_1$ for the $l = 0$ representation. Then the Clebsch-Gordan coefficients are

$$C_{[2],[1],l=0}^{a',b'} = \frac{1}{2}(\delta_{aa'}\delta_{bb'}\delta_{cc'} + \delta_{ab'}\delta_{ba'}\delta_{cc'}) - \frac{1}{8}\delta_{ac}(\delta_{a'c'}\delta_{bb'} + \delta_{b'c'}\delta_{ba'})$$

$$-\frac{1}{8}\delta_{bc}(\delta_{b'c'}\delta_{aa'} + \delta_{a'c'}\delta_{ab'})$$

$$C_{[1,1],[1],l=0}^{a',b'} = \frac{1}{2}(\delta_{aa'}\delta_{bb'}\delta_{cc'} - \delta_{ab'}\delta_{ba'}\delta_{cc'}) - \frac{1}{4}\delta_{ac}(\delta_{a'c'}\delta_{bb'} - \delta_{b'c'}\delta_{ba'})$$

$$-\frac{1}{4}\delta_{bc}(\delta_{b'c'}\delta_{aa'} - \delta_{a'c'}\delta_{ab'})$$

$$C_{l=1,\Lambda_1=([2],[1])}^{a',b'} = \frac{1}{8}\delta_{ac}(\delta_{bb'}\delta_{a'c'} + \delta_{ba'}\delta_{b'c'}) + \frac{1}{8}\delta_{bc}(\delta_{aa'}\delta_{b'c'} + \delta_{ab'}\delta_{a'c'})$$

$$C_{l=1,\Lambda_1=([1,1],[1])}^{a',b'} = \frac{1}{4}\delta_{ac}(\delta_{bb'}\delta_{a'c'} - \delta_{ba'}\delta_{b'c'}) + \frac{1}{4}\delta_{bc}(\delta_{aa'}\delta_{b'c'} - \delta_{ab'}\delta_{a'c'}), \qquad (4.8)$$

giving rise to the following covariant operators

$$O_{[2][1],l=0} = \frac{1}{2}(X_a \otimes X_b \otimes X_c^* + X_b \otimes X_a \otimes X_c^*) - O_{l=1,\Lambda_1=([2][1])}$$

$$O_{[1,1][1],l=0} = \frac{1}{2}(X_a \otimes X_b \otimes X_c^* - X_b \otimes X_a \otimes X_c^*) - O_{l=1,\Lambda_1=([1,1][1])}$$

$$O_{l=1,\Lambda_1=([2][1])} = \frac{1}{8}\delta_{ac}(X_{c'} \otimes X_b \otimes X_{c'}^* + X_b \otimes X_{c'} \otimes X_{c'}^*)$$

$$+\frac{1}{8}\delta_{bc}(X_a \otimes X_{c'} \otimes X_{c'}^* + X_{c'} \otimes X_a \otimes X_{c'}^*)$$

$$O_{l=1,\Lambda_1=([1,1][1])} = \frac{1}{4}\delta_{ac}(X_{c'} \otimes X_b \otimes X_{c'}^* - X_b \otimes X_{c'} \otimes X_{c'}^*)$$

$$+\frac{1}{4}\delta_{bc}(X_a \otimes X_{c'} \otimes X_{c'}^* - X_{c'} \otimes X_a \otimes X_{c'}^*). \tag{4.9}$$

We next present gauge invariant operators. For the $k = 0$ representation of γ, some labels in (3.12) are suppressed to give

$$\mathcal{O}_\Lambda^{\gamma(k=0)} = \sum_{b \in B_N(2,1)} D^\gamma(b^*)tr_{2,1}(bO_\Lambda) = \frac{1}{t^\gamma}tr_{2,1}(P^\gamma O_\Lambda), \tag{4.10}$$

where P^γ is the central projection operator in the Brauer algebra, which is given in [1] by

$$P_{[2][\bar{1}]} = \left(1 - \frac{1}{N+1}C\right)p_{[2]},$$

$$P_{[1^2][\bar{1}]} = \left(1 - \frac{1}{N-1}C\right)p_{[1^2]}. \tag{4.11}$$

Here $C := C_{1\bar{1}} + C_{2\bar{1}}$ commutes with any element in $\mathbb{C}(S_2)$. For the $k = 1$ representation, we have

$$\mathcal{O}_{\Lambda,\Lambda_1}^{\gamma(k=1),A} = \sum_{b \in B_N(2,1)} D_A^\gamma(b^*)tr_{2,1}(bO_{\Lambda,\Lambda_1}) = \frac{1}{t^\gamma}tr_{2,1}(P_A^\gamma O_{\Lambda,\Lambda_1}), \tag{4.12}$$

where

$$P_{[2][\bar{1}]}^{(k=1,\gamma_+=[1],\gamma_-=\emptyset)} = \frac{1}{N+1}Cp_{[2]},$$

$$P_{[1^2][\bar{1}]}^{(k=1,\gamma_+=[1],\gamma_-=\emptyset)} = \frac{1}{N-1}Cp_{[1^2]}. \tag{4.13}$$

We exhibit some of them explicitly

$$tr_{2,1}(P_{[2][\bar{1}]}O_{[2][1],l=0})$$
$$= \frac{1}{2}\left(trX_a trX_b trX_c^\dagger + tr(X_a X_b)trX_c^\dagger\right)$$
$$-\frac{1}{2}\frac{1}{N+1}\left(trX_a tr(X_b X_c^\dagger) + trX_b tr(X_a X_c^\dagger) + tr(X_a X_b X_c^\dagger) + tr(X_b X_a X_c^\dagger)\right)$$
$$-tr_{2,1}(P_{[2][\bar{1}]}O_{l=1,\Lambda_1=[2][1]}),$$
$$tr_{2,1}(P_{[2][\bar{1}]}O_{l=1,\Lambda_1=[2][1]})$$
$$= \frac{1}{8}\delta_{ac}(trX_b trX_{c'} trX_{c'}^* + tr(X_b X_{c'})trX_{c'}^*)$$
$$+\frac{1}{8}\delta_{bc}(trX_a trX_{c'} trX_{c'}^* + tr(X_a X_{c'})trX_{c'}^*)$$
$$-\frac{1}{8}\frac{1}{N+1}\delta_{ac}(trX_b tr(X_{c'} X_{c'}) + trX_{c'} tr(X_b X_{c'}) + tr(X_b X_{c'} X_{c'}^\dagger) + tr(X_b X_{c'}^\dagger X_{c'}))$$
$$-\frac{1}{8}\frac{1}{N+1}\delta_{bc}(trX_a tr(X_{c'} X_{c'}) + trX_{c'} tr(X_a X_{c'}) + tr(X_a X_{c'} X_{c'}^\dagger) + tr(X_a X_{c'}^\dagger X_{c'})).$$

5 Quantum mechanics and conserved charges measuring representation labels

In this section, we study the matrix quantum mechanics which is obtained by the dimensional reduction of the four-dimensional $\mathcal{N} = 4$ SYM on $S^3 \times R$. Because we are mainly interested in the free theory, we ignore the interaction terms. It is found that the Hamiltonian can be given by a set of the harmonic oscillators [5], and it has been known that the harmonic oscillator is characterised by many conserved charges. Some conserved charges whose eigenvalues can specify an orthogonal state were constructed in [3]. Furthermore, the construction was based on symmetries which are enhanced at the free level. In this section, we show conserved charges for the present context.

The Hamiltonian and $U(1)^3$ charge are

$$H = \sum_a tr(A_a^\dagger A_a + B_a^\dagger B_a) + N^2,$$
$$J_a = tr(B_a^\dagger B_a - A_a^\dagger A_a), \tag{5.1}$$

where we have introduced the matrix annihilation and creation operators with the non-zero commutation relations

$$[(A_a)_{ij}, (A_b^\dagger)_{kl}] = \delta_{ab}\delta_{jk}\delta_{il}, \quad [(B_a)_{ij}, (B_b^\dagger)_{kl}] = \delta_{ab}\delta_{jk}\delta_{il}. \tag{5.2}$$

Gauge covariant states are obtained by acting with $A^\dagger$ and $B^\dagger$ on the vacuum with $A|0\rangle = B|0\rangle = 0$. Because the gauge invariance enforces all upper indices to be contracted with all lower indices, all gauge invariant operators take the form of $tr_{m,n}(\Sigma(b)(A^\dagger)^{\otimes m} \otimes (B^\dagger)^{\otimes n})$. Here Σ is a map from elements in $B_N(m,n)$ to elements in $\mathbb{C}(S_{m+n})$ introduced in [1].[8] Our claim in the previous sections can be straightforwardly applied to give the following orthogonal state,

$$|\mathcal{O}_{\Lambda,M_\Lambda,\Lambda_1,\tau}^{\gamma,A,ij,\tau_{\Lambda_1}}\rangle = \sum_{m_A,m_A',m_{\Lambda_1}} C_{A,m_A,A,m_A'}^{\tau_{\Lambda_1},\Lambda_1,m_{\Lambda_1}} \sum_b D_{A,m_A,j,m_A',i}^{\gamma}(b^*) \sum_{\vec{a},\vec{b}} C_{\Lambda,M_\Lambda,\Lambda_1,m_{\Lambda_1},\tau}^{\vec{a},\vec{b}} tr_{m,n}(\Sigma(b)O_{\vec{a},\vec{b}})|0\rangle, \tag{5.3}$$

where

$$O_{\vec{a},\vec{b}} = A_{a_1}^\dagger \otimes \cdots \otimes A_{a_m}^\dagger \otimes B_{b_1}^\dagger \otimes \cdots \otimes B_{b_n}^\dagger. \tag{5.4}$$

The inner product is diagonal:

$$\langle \mathcal{O}_{\Lambda',M_{\Lambda'},\Lambda_1',\tau'}^{\gamma',A',i'j',\tau_{\Lambda_1}'} | \mathcal{O}_{\Lambda,M_\Lambda,\Lambda_1,\tau}^{\gamma,A,ij,\tau_{\Lambda_1}} \rangle = m!n!d_{\Lambda_1} \frac{1}{t^\gamma} \delta_{\gamma\gamma'}\delta_{AA'}\delta_{\Lambda\Lambda'}\delta_{M_\Lambda M_{\Lambda'}}\delta_{\Lambda_1\Lambda_1'}\delta_{\tau_{\Lambda_1}\tau_{\Lambda_1'}}\delta_{\tau\tau'}\delta_{ii'}\delta_{jj'}. \tag{5.5}$$

Before showing a set of conserved charges whose eigenvalues identify the labels of the orthogonal state, we shall explain symmetries of the Hamiltonian.

---

[8]One simple example is $tr(XX^\dagger) = tr_{1,1}(CX \otimes X^*) = tr_2(\Sigma(C)X \otimes X^\dagger)$. The map Σ was also exploited in [14] to construct another non-holomorphic extension for a single complex matrix.

The original scalar matrix field theory action is invariant under the adjoint unitary transformation. We can express the conserved matrix current in terms of the annihilation and creation operators as $[A_a, A_a^\dagger]$ for the A-sector and $[B_a, B_a^\dagger]$ for the B-sector. We will find that we have more symmetries at the free level. In order to see enhanced symmetries of this Hamiltonian, we decompose[9] the generator and name them as

$$
\begin{aligned}
(G_{L,A})_{ij} &:= \sum_a (G_{L,A_a})_{ij}, & (G_{L,A_a})_{ij} &:= (A_a^\dagger)_{kj}(A_a)_{ik} \\
(G_{R,A})_{ij} &:= \sum_a (G_{R,A_a})_{ij}, & (G_{R,A_a})_{ij} &:= (A_a^\dagger)_{ik}(A_a)_{kj}
\end{aligned}
\tag{5.6}
$$

and similarly for the B-sector. We also define

$$
\begin{aligned}
(G_L)_{ij} &:= (G_{L,A})_{ij} + (G_{L,B})_{ij}, \\
(G_R)_{ij} &:= (G_{R,A})_{ij} + (G_{R,B})_{ij}.
\end{aligned}
\tag{5.7}
$$

They generate the left action and the right action:

$$
\begin{aligned}
[tr(\Lambda G_{L,A}), (A_a)_{ij}] &= -(\Lambda A_a)_{ij}, \\
[tr(\Lambda G_{R,A}), (A_a)_{ij}] &= (A_a \Lambda)_{ij}.
\end{aligned}
\tag{5.8}
$$

It is easy to show that

$$
[G_{L,A}, tr(A_a A_a^\dagger)] = 0, \quad [G_{R,A}, tr(A_a A_a^\dagger)] = 0,
\tag{5.9}
$$

and similar equations for the B-sector, implying that all of $G_{L,A}$, $G_{R,A}$ and $G_{L,B}$, $G_{R,B}$ generate symmetries of the Hamiltonian. They form the $u(N)$ commutation relations

$$
\begin{aligned}
[(G_{L,A_a})_{ij}, (G_{R,A_a})_{kl}] &= 0, \\
[(G_{L,A_a})_{ij}, (G_{L,A_b})_{kl}] &= \delta_{ab}((G_{L,A_a})_{kj}\delta_{il} - (G_{L,A_a})_{il}\delta_{jk}), \\
[(G_{R,A_a})_{ij}, (G_{R,A_b})_{kl}] &= \delta_{ab}((G_{R,A_a})_{il}\delta_{jk} - (G_{R,A_a})_{kj}\delta_{il}).
\end{aligned}
\tag{5.10}
$$

Since we also have the same relations for the B-sector, we thus have four commuting copies of the $u(N)$ algebra.

In terms of these generators, the Hamiltonian and the angular momentum can be expressed by

$$
\begin{aligned}
H &= tr(G_{L,A} + G_{L,B}), \\
J_a &= tr(G_{L,B_a} - G_{L,A_a}).
\end{aligned}
\tag{5.11}
$$

It is mentioned that $tr(G_{L,A_a}) = tr(G_{R,A_a})$ and $tr(G_{L,B_a}) = tr(G_{R,B_a})$. Because $G_L + G_R$ generates the adjoint gauge transformation, we have

$$
G_L + G_R = 0
\tag{5.12}
$$

on gauge invariant states.

---

[9]Such decomposition was also considered in [23].

From now on we present some conserved operators which act on the orthogonal state with eigenvalues measuring the representation labels. We shall build operators from the symmetry generators, so it is manifest that they commute with the Hamiltonian. The construction of those operators is shown in [3]. We define the symbol $\doteq$ to assume the actions on $|\mathcal{O}^{\gamma,A,ij,\tau_{\Lambda_1}}_{\Lambda,M_\Lambda,\Lambda_1,\tau}\rangle$. We first show three operators

$$
\begin{aligned}
tr(G_L)^2 &\doteq tr(G_R)^2 \doteq C_2(\gamma), \\
tr\,(G_{L,A})^2 &\doteq tr\,(G_{R,A})^2 \doteq C_2(\alpha), \\
tr\,(G_{L,B})^2 &\doteq tr\,(G_{R,B})^2 \doteq C_2(\beta),
\end{aligned}
\tag{5.13}
$$

where C_2 is the quadratic Casimir of U(N). These commute each other without assuming the action on the state. We also have similar equations for higher order actions, for example,

$$
tr\,(G_L)^r \doteq tr\,(G_R)^r \doteq C_r(\gamma).
\tag{5.14}
$$

It was also shown that the multiplicity associated with the decomposition $\gamma \to A$ of the colour structure can be measured as

$$
\begin{aligned}
tr\left((G_{L,A})^2 G_{L,B}\right) &\doteq C(\gamma,\alpha,\beta,i), \\
tr\left((G_{R,A})^2 G_{R,B}\right) &\doteq C(\gamma,\alpha,\beta,j),
\end{aligned}
\tag{5.15}
$$

where $C(\gamma,\alpha,\beta,i)$ is a quantity depending on γ, α, β, i, but the exact form has not been found.[10]

In addition to the enhanced symmetry generators $G_{L,A}$, $G_{R,A}$ and $G_{L,B}$, $G_{R,B}$, we have another enhanced symmetry which is generated by

$$
(G_E)_{kljm} := (G_{E,A})_{kljm} + (G_{E,B})_{kljm},
\tag{5.16}
$$

where

$$
(G_{E,A})_{kljm} := \sum_a (G_{E,A_a})_{kljm} := \sum_a (A_a)^\dagger_{kl}(A_a)_{jm}.
\tag{5.17}
$$

We find that this generates a U(N^2) symmetry, which can be manifested by introducing a composite index $I = (i,j)$, where I takes N^2 values when i and j run over N values. The use of the composite index enables us to express the generator as

$$
(G_{E,A})_{IJ} = \sum_a (A_a)^\dagger_I (A_a)_J.
\tag{5.18}
$$

The commutation relation of G_E is

$$
[(G_E)_{IJ},(G_E)_{KL}] = (G_E)_{IL}\delta_{JK} - (G_E)_{KJ}\delta_{IL}.
\tag{5.19}
$$

The actions of $G_{E,A}$ on A_a and $A^\dagger_a$ are

$$
\begin{aligned}
[(G_{E,A})_{IJ},(A^\dagger_b)_K] &= \delta_{JK}(A^\dagger_b)_I, \\
[(G_{E,A})_{IJ},(A_b)_K] &= -\delta_{IK}(A_b)_J,
\end{aligned}
\tag{5.20}
$$

---

[10]In the paper [3], it was shown that $tr\left((G_{L,A})^2 G_{L,B}\right)$ can recognise the multiplicity index, and it was confirmed for some examples.

where we have defined $\delta_{JK} = \delta_{j_1 k_2} \delta_{j_2 k_1}$ for $J = (j_1, j_2)$ and $K = (k_1, k_2)$. We also have

$$[(G_{E,A})_{kljm}, (G_{L,A})_{pq}] = (G_{E,A})_{klpm}\delta_{jq} - (G_{E,A})_{kqjm}\delta_{lp},$$
$$[(G_{E,A})_{kljm}, (G_{R,A})_{pq}] = (G_{E,A})_{kljq}\delta_{pm} - (G_{E,A})_{pljm}\delta_{qk}. \tag{5.21}$$

It is also easy to see that the symmetry generated by G_E is indeed a symmetry of the Hamiltonian:

$$[(G_{E,A})_{kljm}, tr(A_a^\dagger A_a)] = 0. \tag{5.22}$$

With the enhanced $U(N^2)$ symmetry we can define an operator which can measure τ_1, i.e. the number of copies of Λ_1 in the inner tensor product of $A \otimes A$, as

$$tr\,(G_L G_E G_R) \doteq C(\tau_{\Lambda_1}). \tag{5.23}$$

$C(\tau_{\Lambda_1})$ is a quantity depending on τ_1.

The representation labels associated with the flavour indices can be measured in terms of $E_{Aab} := tr(A_a^\dagger A_b)$, whose action on $A^\dagger$ is

$$[E_{Aab}, (A_c^\dagger)_{ij}] = \delta_{bc}(A_a^\dagger)_{ij}. \tag{5.24}$$

E_{Aab} satisfies the $u(p)$ commutation relation:

$$[E_{Aab}, E_{Acd}] = \delta_{bc}E_{Aad} - \delta_{ad}E_{Acb}. \tag{5.25}$$

The quadratic Casimir of $u(p)$ appears an eigenvalue of the quadratic action of $E_{ab} := E_{Aab} + E_{Bab}$ on the orthogonal state:

$$E_{ab}E_{ab} \doteq C_2(\Lambda). \tag{5.26}$$

We also have

$$E_{Aab}E_{Aab} \doteq C_2(\alpha_1), \quad E_{Bab}E_{Bab} \doteq C_2(\beta_1) \tag{5.27}$$

for $\Lambda_1 = (\alpha_1, \beta_1)$. The index τ which runs over the multiplicity of Λ_1 in Λ would be measured by

$$E_{Aab}E_{Abc}E_{Bca}. \tag{5.28}$$

In summary, we have provided some conserved operators which can be diagonalised by the orthogonal state. In other words, the orthogonal state proposed in this paper can be specified by the simultaneous diagonalisation of the conserved operators.

Operators which have the diagonal actions on an orthogonal state can be organised by the enhanced symmetries of the Hamiltonian. General multi-trace operators made from those generators are all conserved, and they would form an extension of the W_∞ algebra. It is interesting to ask how the enhanced symmetries can be connected with the integrability of this system. Conserved charges may also be helpful to understand gravitational duals or space-time interpretations (see [24, 25] for such discussions).

6 Discussions

In this paper, we have proposed an orthogonal set of non-holomorphic gauge invariant operators made from some complex matrices at the free coupling based on the Brauer algebra. Below are possible future directions along this line.

One remaining problem which should be discussed is to see if this proposed operator exhausts multi-matrix gauge invariant operators. For the case of a single complex matrix, a counting formula in terms of group theoretic languages was given in [1], and it is proved for large N in [26]. It has not, however, been cleared how the counting is modified when N is finite in the context of Brauer algebra. One modification of the finite N case is to impose the constraint $c_1(\gamma_+) + c_1(\gamma_-) \leq N$, which is naturally expected from the definition of the U(N) group. This constraint seems to hold the attention because it realises a generalisation of the cut-off $c_1(R) \leq N$ for the half-BPS case. In the half-BPS case, the cut-off can be translated into the cut-off for the angular momentum of giant gravitons [10]. (Such an effect was originally studied in [27, 28].) The cut-off for the present case would give a constraint for a composite angular momentum of giant gravitons and anti-giant gravitons. For the single complex matrix case, more analyses at finite N will be given in [26]. The counting operators including the finite N case for the non-holomorphic multi-matrix case will be discussed in future publications.

Another basis of non-holomorphic one-matrix gauge invariant operators was built in [14], where the symmetric group plays a role instead of the Brauer algebra. This basis was originally proposed to study excitations of giant gravitons [29–32]. It will be possible to construct another basis for non-holomorphic multi-matrix operators as an extension of [14]. As is discussed in [3], a difference of two bases in a sector can be explained by the fact that a set of Casimirs characterising an orthogonal basis does not commute with another set of Casimirs characterising another orthogonal basis. It would be nice to ask roles of two different bases from the point of view of dual physics.

For the highest weight state in the half-BPS sector, the physics is characterised by a single Young diagram alone. The number and the angular momentum of giant gravitons are encrypted in a single Young diagram. On the other hand, the Brauer algebra brings in two kinds of representation labels, i.e. $\gamma = (\gamma_+, \gamma_-, k)$, and $A = (\alpha, \beta)$. Having diagonalised two-point functions at the free level, the representation labels will be directly related to a tensionless string theory. In our previous paper [1], it was conjectured that the operator has a close connection with a system of giant gravitons and anti-giant gravitons. The $k = 0$ representation would be naturally related to such a system, where $\gamma_+ = \alpha$ would read the number and angular momentum of giant gravitons while $\gamma_- = \beta$ would read those of anti-giant gravitons. Because k is the number of boxes which are got rid of from m boxes and n anti-boxes, one can anticipate that $k \neq 0$ representations would contain branes and anti-branes with smaller quantum numbers. At $k = m = n$, one may expect closed string excitations without branes and anti-branes. If we include corrections of the t'Hooft coupling, this system may start showing the instability originated from the existence of branes and anti-branes. If our operators can really describe physics of the unstable system, it is natural to expect physics of tachyon condensation to be encoded into the representation

labels in a way. Fortunately, it is possible to study this system at finite string scale where tachyon has the negative mass squared because this SO(6) sector is closed at one-loop. In this sense, this system will be a good framework to know how the Brauer algebra captures such an interesting unstable system. The operator mixing problem has been reported in [33–37]. On the other hand, the one-loop correction was studied in the language of the representation basis for the U(2) holomorphic sector in [39], where a restricted mixing pattern was found for Young diagrams. A similar restricted mixing was also given in [31, 32] in the context of the restricted Schur polynomials. The mixing problem was studied in [38] in terms of a basis expressed by the symmetric group, and the Hamiltonian was expressed by splitting and joining interactions. Studies along these lines using the bases proposed in this paper would shed light on the role of the Brauer algebra.

Our orthogonal operators are entirely specified by some group theoretic operators constructed from generators for the enhanced symmetries. Because Casimir operators know information about representation labels of orthogonal sets, studies of Casimirs operators at the one-loop level would tell us about the operator mixing. For example, a relation between the one-loop dilatation operator and the Casimirs may suggest how orthogonal operators mix under quantum corrections. The breaking of the enhanced symmetries with the interactions turned on can be possibly associated with the operator mixing problem. We hope to understand the role of the enhanced symmetries by looking for a connection to the integrability.

Acknowledgments

A   One-loop dilatation operator

In this section, we rewrite the one-loop dilatation operator in terms of the complex variables.

The dilatation operator in the SO(6) sector up to one-loop order was given [40, 41] by

$$\mathcal{D} = \mathcal{D}_0 + \mathcal{D}_2 := \mathcal{D}_0 + g^2\mathcal{H}, \qquad (g^2 = g_{\mathrm{YM}}^2 N/8\pi^2) \tag{A.1}$$

where

$$\mathcal{H} = N^{-1}\left(-\frac{1}{2} : tr[\Phi_m, \Phi_n][\check{\Phi}^m, \check{\Phi}^n] : -\frac{1}{4} : tr[\Phi_m, \check{\Phi}^n][\Phi_m, \check{\Phi}^n] : \right). \tag{A.2}$$

Φ_m $(m = 1, \ldots, 6)$ is the scalar fields. $\check{\Phi}$ represents the following derivative action

$$(\check{\Phi}^m)_{ij}(\Phi_n)_{kl} = \delta_n^m \delta_{il}\delta_{kj} \tag{A.3}$$

for the $u(N)$ gauge group.

We define the complex combination of two scalars

$$X_a = \Phi_{2a-1} + i\Phi_{2a}, \quad (a = 1, 2, 3). \tag{A.4}$$

Solving for Φ, we get

$$\Phi_{2a-1} = \frac{1}{2}\left(X_a + X_a^\dagger\right), \quad \Phi_{2a} = \frac{1}{2i}\left(X_a - X_a^\dagger\right). \tag{A.5}$$

We express these equations for later convenience as

$$\Phi_m = a_m^i Z_i, \tag{A.6}$$

where $Z_i = (X_1, X_1^\dagger, X_2, X_2^\dagger, X_3, X_3^\dagger)$. The derivative of the complex matrix is defined by

$$\check{X}^a = \frac{1}{2}\left(\check{\Phi}^{2a-1} - i\check{\Phi}^{2a}\right), \tag{A.7}$$

satisfying $(\check{X}^a)_{ij}(X_b)_{kl} = \delta_{ab}\delta_{il}\delta_{kl}$. We solve for $\check{\Phi}$ as

$$\check{\Phi}_{2a-1} = \check{X}_a + \check{X}_a^\dagger, \quad \check{\Phi}_{2a} = i\left(\check{X}_a - \check{X}_a^\dagger\right), \tag{A.8}$$

which we shall denote compactly by

$$\check{\Phi}_m = b_m^i \check{Z}_i, \tag{A.9}$$

where $\check{Z}_i = (\check{X}_1, \check{X}_1^\dagger, \check{X}_2, \check{X}_2^\dagger, \check{X}_3, \check{X}_3^\dagger)$.

Using these complex variables, we can rewrite the first term of the Hamiltonian (A.2) as

$$
\begin{aligned}
tr[\Phi_m, \Phi_n][\check{\Phi}^m, \check{\Phi}^n] &= a_m^i a_n^j b_m^k b_n^l tr[Z_i, Z_j][\check{Z}^k, \check{Z}^l] \\
&= g^{ik} g^{jl} tr[Z_i, Z_j][\check{Z}^k, \check{Z}^l] \\
&= tr[X_a, X_b][\check{X}_a, \check{X}_b] + tr[X_a, X_b^\dagger][\check{X}_a, \check{X}_b^\dagger] \\
&\quad + tr[X_a^\dagger, X_b][\check{X}_a^\dagger, \check{X}_b] + tr[X_a^\dagger, X_b^\dagger][\check{X}_a^\dagger, \check{X}_b^\dagger].
\end{aligned} \tag{A.10}
$$

We have defined the metric $g^{ik} = \sum_m a_m^i b_m^k = \delta^{ik}$. On the other hand, the second term of the Hamiltonian can be rewritten as

$$
\begin{aligned}
tr[\Phi_m, \check{\Phi}^n][\Phi_m, \check{\Phi}^n] &= a_m^i b_n^j a_m^k b_n^l tr[Z_i, \check{Z}^j][Z_k, \check{Z}^l] \\
&= h^{ik} \tilde{h}^{jl} tr[Z_i, \check{Z}^j][Z_k, \check{Z}^l] \\
&= tr[X_a, \check{X}_b][X_a^\dagger, \check{X}_b^\dagger] + tr[X_a, \check{X}_b^\dagger][X_a^\dagger, \check{X}_b] \\
&\quad + tr[X_a^\dagger, \check{X}_b^\dagger][X_a, \check{X}_b] + tr[X_a^\dagger, \check{X}_b][X_a, \check{X}_b^\dagger] \\
&= 2tr[X_a, \check{X}_b][X_a^\dagger, \check{X}_b^\dagger] + 2tr[X_a, \check{X}_b^\dagger][X_a^\dagger, \check{X}_b],
\end{aligned} \tag{A.11}
$$

where we have defined the metric as $h^{ik} = a_m^i a_m^k$ and $\tilde{h}^{jl} = b_n^j b_n^l$, with non-zero components $h^{X_a \bar{X}_a} = 1/2$ and $\tilde{h}^{X_a \bar{X}_a} = 2$. Collecting (A.10) and (A.11), we get the Hamiltonian in terms of the complex variables as

$$
\begin{aligned}
\mathcal{H} = -\frac{1}{2N} : \Big(&tr[X_a, X_b][\check{X}_a, \check{X}_b] + tr[X_a, X_b^\dagger][\check{X}_a, \check{X}_b^\dagger] \\
&+ tr[X_a^\dagger, X_b][\check{X}_a^\dagger, \check{X}_b] + tr[X_a^\dagger, X_b^\dagger][\check{X}_a^\dagger, \check{X}_b^\dagger] \\
&+ tr[X_a, \check{X}_b][X_a^\dagger, \check{X}_b^\dagger] + tr[X_a, \check{X}_b^\dagger][X_a^\dagger, \check{X}_b] \Big) : .
\end{aligned} \tag{A.12}
$$

B Proof of the diagonal two-point function

In this section, we show detailed calculations of the two-point functions.

B.1 Two-point function for the covariant operator

The two-point function of the representation basis (3.6) can be computed as

$$
\begin{aligned}
&\langle : (O^{\dagger}_{\Lambda,M_\Lambda,\Lambda_1,m_{\Lambda_1},\tau})^I_J :: (O_{\Lambda',M'_{\Lambda'},\Lambda'_1,m'_{\Lambda'_1},\tau'})^K_L : \rangle \\
&= \sum_{a,b,a',b'} (C^{\vec{a},\vec{b}}_{\Lambda,M_\Lambda,\Lambda_1,m_{\Lambda_1},\tau})^* C^{\vec{a'},\vec{b'}}_{\Lambda',M'_{\Lambda'},\Lambda'_1,m'_{\Lambda'_1},\tau'} \langle : (O^{\dagger}_{\vec{a},\vec{b}})^I_J :: (O_{\vec{a'},\vec{b'}})^K_L : \rangle \\
&= \sum_{a,b,a',b'} (C^{\vec{a},\vec{b}}_{\Lambda,M_\Lambda,\Lambda_1,m_{\Lambda_1},\tau})^* C^{\vec{a'},\vec{b'}}_{\Lambda',M'_{\Lambda'},\Lambda'_1,m'_{\Lambda'_1},\tau'} \sum_{\sigma \in S_m \times S_n} \prod_{k=1}^m \delta_{a_k a'_{\sigma(k)}} \prod_{l=1}^n \delta_{b_l b'_{\sigma(l)}} (\sigma)^K_J (\sigma^{-1})^I_L \\
&= \sum_{a,b} \sum_{\sigma \in S_m \times S_n} (C^{\vec{a},\vec{b}}_{\Lambda,M_\Lambda,\Lambda_1,m_{\Lambda_1},\tau})^* C^{a_{\sigma^{-1}(1)},\ldots,a_{\sigma^{-1}(m)},b_{\sigma^{-1}(1)},\ldots,a_{\sigma^{-1}(n)}}_{\Lambda',M'_{\Lambda'},\Lambda'_1,m'_{\Lambda'_1},\tau'} (\sigma)^K_J (\sigma^{-1})^I_L \\
&= \delta_{\Lambda\Lambda'} \delta_{M_\Lambda M'_{\Lambda'}} \delta_{\Lambda_1 \Lambda'_1} \delta_{\tau\tau'} \sum_{\sigma \in S_m \times S_n} D^{\Lambda_1}_{m_{\Lambda_1} m'_{\Lambda'_1}}(\sigma) (\sigma)^K_J (\sigma^{-1})^I_L . \quad\text{(B.1)}
\end{aligned}
$$

To get the third equality, we solved the delta symbols. In the last step, we have used

$$
C^{a_{\sigma^{-1}(1)},\ldots,a_{\sigma^{-1}(m)},b_{\sigma^{-1}(1)},\ldots,a_{\sigma^{-1}(n)}}_{\Lambda',M'_{\Lambda'},\Lambda'_1,m'_{\Lambda'_1},\tau'} = D^{\Lambda_1}_{m'_{\Lambda'_1} m''_{\Lambda'_1}}(\sigma^{-1}) C^{a_1,\ldots,a_m,b_1,\ldots,a_n}_{\Lambda',M'_{\Lambda'},\Lambda'_1,m''_{\Lambda'_1},\tau'} \quad\text{(B.2)}
$$

and the following equation

$$
\sum_{\vec{a},\vec{b}} (C^{\Lambda,M_\Lambda,\Lambda_1,m_{\Lambda_1},\tau}_{\vec{a},\vec{b}})^* C^{\Lambda',M'_{\Lambda'},\Lambda'_1,m'_{\Lambda'_1},\tau'}_{\vec{a},\vec{b}} = \delta_{\Lambda\Lambda'} \delta_{M_\Lambda M'_{\Lambda'}} \delta_{\Lambda_1 \Lambda'_1} \delta_{m_{\Lambda_1} m'_{\Lambda'_1}} \delta_{\tau\tau'} . \quad\text{(B.3)}
$$

The derivation of this equation is completely similar to (69) in [17]. (B.2) comes from the fact that σ^{-1} acts only on $V^{\mathbb{C}(S_m \times S_n)}_{\Lambda_1}$ when it acts on $V^{\otimes m} \otimes \bar{V}^{\otimes n}$ in (3.4).

B.2 Two-point function for the gauge invariant operator

In this subsection, we shall present the proof of (3.15). Using the two-point function of the covariant operator (3.10), we get

$$
\begin{aligned}
&\langle \mathcal{O}^{\gamma,A,ij,\tau_{\Lambda_1}\,\dagger}_{\Lambda,M_\Lambda,\Lambda_1,\tau} \mathcal{O}^{\gamma',A',i'j',\tau'_{\Lambda'_1}}_{\Lambda',M'_{\Lambda'},\Lambda'_1,\tau'} \rangle \\
&= \delta_{\Lambda\Lambda'} \delta_{M_\Lambda M'_{\Lambda'}} \delta_{\Lambda_1 \Lambda'_1} \delta_{\tau\tau'} C^{\tau_{\Lambda_1},\Lambda_1,m_{\Lambda_1}}_{A,m_A,A,m'_A} C^{\tau'_{\Lambda'_1},\Lambda'_1,m'_{\Lambda'_1}}_{A',m_{A'},A',m'_{A'}} \\
&\quad \times \sum_{\sigma \in S_m \times S_n} D^{\Lambda_1}_{m_{\Lambda_1} m'_{\Lambda'_1}}(\sigma) \sum_{b,b' \in B_N(m,n)} D^{\gamma}_{A,m_A j,m'_A i}(b^*) D^{\gamma'}_{A',m_{A'}j',m'_{A'}i'}(b'^*) tr_{m,n}(b^{\dagger}\sigma b' \sigma^{-1}), \quad\text{(B.4)}
\end{aligned}
$$

where we have used the fact that the Clebsch-Gordan coefficient $C^{\tau_{\Lambda_1},\Lambda_1,m_{\Lambda_1}}_{A,m_A,A,m'_A}$ and the representation matrix of Brauer elements are real.

Let us work on $tr_{m,n}(b\sigma b'\sigma^{-1})$. Using the Schur-Weyl duality (2.6), we find

$$
tr_{m,n}(b^{\dagger}\sigma b'\sigma^{-1}) = \sum_{\gamma''} t^{\gamma''} \chi^{\gamma''}(b^{\dagger}\sigma b'\sigma^{-1}), \quad\text{(B.5)}
$$

where $t^{\gamma''}$ and $\chi^{\gamma''}(b)$ are the dimension of U(N) and the character of the Brauer algebra associated with the irreducible representation γ''. We decompose the character by inserting the completeness of $|\gamma \to A, m_A, i_A\rangle$ as

$$\chi^{\gamma''}(b^\dagger \sigma b' \sigma^{-1}) = \langle \gamma'' \to B, m_B, i_B | b^\dagger | \gamma'' \to C, m_C, i_C \rangle D^C_{m_C m'_C}(\sigma)$$
$$\langle \gamma'' \to C, m'_C, i_C | b' | \gamma'' \to B, m'_B, i_B \rangle D^B_{m'_B m_B}(\sigma^{-1}), \qquad (\text{B.6})$$

where the summation is assumed to be taken for all repeated letters except γ''. We note that

$$\langle \gamma'' \to B, m_B, i_B | \sigma | \gamma'' \to C, m_C, i_C \rangle = D^B_{m_B m_C}(\sigma) \delta_{BC} \delta_{i_B i_C}. \qquad (\text{B.7})$$

Using the above decomposition of the character, one can calculate (B.4) as

$$\langle \mathcal{O}^{\gamma, A, ij, \tau_{\Lambda_1}}_{\Lambda, M_\Lambda, \Lambda_1, \tau} {}^\dagger \mathcal{O}^{\gamma', A', i'j', \tau'_{\Lambda'_1}}_{\Lambda', M'_{\Lambda'}, \Lambda'_1, \tau'} \rangle$$
$$= \delta_{\Lambda \Lambda'} \delta_{M_\Lambda M'_{\Lambda'}} \delta_{\Lambda_1 \Lambda'_1} \delta_{\tau \tau'} C^{\tau_{\Lambda_1}, \Lambda_1, m_{\Lambda_1}}_{A, m_A, A, m'_A} C^{\tau'_{\Lambda'_1}, \Lambda'_1, m'_{\Lambda'_1}}_{A', m_{A'}, A', m'_{A'}}$$
$$\times \sum_{\sigma \in S_m \times S_n} D^{\Lambda_1}_{m_{\Lambda_1} m'_{\Lambda'_1}}(\sigma) \sum_{b, b'} D^\gamma_{A, m_A j, m'_A i}(b^*) D^{\gamma'}_{A', m_{A'} j', m'_{A'} i'}(b'^*)$$
$$\times t^{\gamma''} \langle \gamma'' \to B, m_B, i_B | b^\dagger | \gamma'' \to C, m_C, i_C \rangle D^C_{m_C m'_C}(\sigma)$$
$$\times \langle \gamma'' \to C, m'_C, i_C | b' | \gamma'' \to B, m'_B, i_B \rangle D^B_{m'_B m_B}(\sigma^{-1})$$
$$= \delta_{\Lambda \Lambda'} \delta_{M_\Lambda M'_{\Lambda'}} \delta_{\Lambda_1 \Lambda'_1} \delta_{\tau \tau'} C^{\tau_{\Lambda_1}, \Lambda_1, m_{\Lambda_1}}_{A, m_A, A, m'_A} C^{\tau'_{\Lambda'_1}, \Lambda'_1, m'_{\Lambda'_1}}_{A', m_{A'}, A', m'_{A'}}$$
$$\times \frac{1}{t^\gamma} \delta_{\gamma \gamma'} \delta_{AA'} \sum_{\sigma \in S_m \times S_n} D^{\Lambda_1}_{m_{\Lambda_1} m'_{\Lambda'_1}}(\sigma) D^A_{m'_A m'_{A'}}(\sigma) D^A_{m_{A'} m_A}(\sigma^{-1}) \delta_{i'i} \delta_{jj'}$$
$$= \delta_{\Lambda \Lambda'} \delta_{M_\Lambda M'_{\Lambda'}} \delta_{\Lambda_1 \Lambda'_1} \delta_{\tau \tau'} \delta_{AA'} C^{\tau_{\Lambda_1}, \Lambda_1, m_{\Lambda_1}}_{A, m_A, A, m'_A} C^{\tau'_{\Lambda'_1}, \Lambda'_1, m'_{\Lambda'_1}}_{A', m_{A'}, A', m'_{A'}}$$
$$\times \frac{1}{t^\gamma} \delta_{\gamma \gamma'} \frac{m! n!}{d_{\Lambda_1}} \sum_{\tau'_{\Lambda_1}} \delta_{ii'} \delta_{jj'} C^{\tau'_{\Lambda_1}, \Lambda_1, m_{\Lambda_1}}_{A, m'_A, A, m_A} C^{\tau'_{\Lambda_1}, \Lambda_1, m'_{\Lambda'_1}}_{A, m'_{A'}, A, m_{A'}}$$
$$= \delta_{\Lambda \Lambda'} \delta_{M_\Lambda M'_{\Lambda'}} \delta_{\Lambda_1 \Lambda'_1} \delta_{\tau \tau'} \delta_{AA'} \delta_{\tau_{\Lambda_1} \tau'_{\Lambda'_1}} \delta_{ii'} \delta_{jj'} m! n! d_{\Lambda_1} \frac{1}{t^\gamma} \delta_{\gamma \gamma'}. \qquad (\text{B.8})$$

The second equality was obtained by performing the summation for b and b' using the formula derived in subsection B.3. To show the third equality, we have used

$$\sum_{\sigma \in S_m \times S_n} D^{\Lambda_1}_{m_{\Lambda_1} m'_{\Lambda'_1}}(\sigma) D^A_{m'_A m'_{A'}}(\sigma) D^A_{m_{A'} m_A}(\sigma^{-1})$$
$$= \sum_{\sigma \in S_m \times S_n} D^{\Lambda_1}_{m_{\Lambda_1} m'_{\Lambda'_1}}(\sigma) D^A_{m'_A m'_{A'}}(\sigma) D^A_{m_A m_{A'}}(\sigma)$$
$$= \frac{m! n!}{d_{\Lambda_1}} \sum_{\tau'_{\Lambda_1}} C^{\tau'_{\Lambda_1}, \Lambda_1, m_{\Lambda_1}}_{A, m'_A, A, m_A} C^{\tau'_{\Lambda_1}, \Lambda_1, m'_{\Lambda'_1}}_{A, m'_{A'}, A, m_{A'}}. \qquad (\text{B.9})$$

This is a generalisation of (163) in [13]. The forth equality of (B.8) follows from the orthogonality of Clebsch-Gordan coefficient [22]:

$$\sum_{m_A, m'_A} C^{\tau_{\Lambda_1}, \Lambda_1, m_{\Lambda_1}}_{A, m_A, A, m'_A} C^{\tau'_{\Lambda'_1}, \Lambda'_1, m'_{\Lambda'_1}}_{A, m_A, A, m'_A} = \delta_{\tau_{\Lambda_1}, \tau'_{\Lambda'_1}} \delta_{\Lambda_1, \Lambda'_1} \delta_{m_{\Lambda_1}, m'_{\Lambda'_1}}. \qquad (\text{B.10})$$

B.3 A formula

We now calculate

$$\sum_{b\in B_N(m,n)} \langle \gamma \to A, m_A, i_A | b^* | \gamma \to B, m_B, i_B \rangle \langle \gamma' \to C, m_C, i_C | b | \gamma' \to D, m_D, i_D \rangle. \quad \text{(B.11)}$$

This can be worked out as

$$\sum_{b\in B_N(m,n)} \sum_{I,J,K,L} B^\dagger_{\gamma I;A,m_A,i_A} B_{\gamma J;B,m_B,i_B} D^\gamma_{IJ}(b^*) B^\dagger_{\gamma'K;C,m_C,i_C} B_{\gamma'L;D,m_D,i_D} D^{\gamma'}_{KL}(b)$$
$$= \frac{1}{t^\gamma} \delta_{\gamma\gamma'} \sum_{I,J} B^\dagger_{\gamma I;A,m_A,i_A} B_{\gamma J;B,m_B,i_B} B^\dagger_{\gamma J;C,m_C,i_C} B_{\gamma I;D,m_D,i_D}$$
$$= \frac{1}{t^\gamma} \delta_{\gamma\gamma'} \delta_{AD} \delta_{BC} \delta_{m_A m_D} \delta_{m_B m_C} \delta_{i_A i_D} \delta_{i_B i_C}. \quad \text{(B.12)}$$

Some equations needed to show this are collected in the following.

- A representation matrix for an element b in the Brauer algebra can be denoted by $D^\gamma_{IJ}(b) = \langle \gamma, I | b | \gamma, j \rangle$ ($1 \le I, J \le d_\gamma$). d_γ is the dimension of the Brauer algebra relevant to the irreducible representation γ. The basis is complete:

$$\sum_{I=1}^{d_\gamma} | \gamma, I \rangle \langle \gamma, I | = 1. \quad \text{(B.13)}$$

  The orthogonality of the representation matrix is [42]

$$\sum_{b\in B_N(m,n)} D^\gamma_{IJ}(b^*) D^{\gamma'}_{KL}(b) = \frac{1}{t^\gamma} \delta_{JK} \delta_{IL} \delta^{\gamma\gamma'}. \quad \text{(B.14)}$$

- We introduce the branching coefficient $B_{\gamma I;A,m_A,i}$ as

$$B_{\gamma I;A,m_A,i} = \langle \gamma, I | \gamma \to A, m_A, i \rangle. \quad \text{(B.15)}$$

  The orthogonality of the branching coefficient can be derived as

$$\sum_I (B_{\gamma,I;A,m_A,i_A})^\dagger B_{\gamma,I;B,m_B,i_B} = \sum_I \langle \gamma \to A, m_A, i_A | \gamma, I \rangle \langle \gamma, I | \gamma \to B, m_B, i_B \rangle$$
$$= \delta_{AB} \delta_{m_A m_B} \delta_{i_A i_B}, \quad \text{(B.16)}$$

  where we have chosen an orthogonal basis

$$\langle \gamma \to B, m_B, k | \gamma \to A, m_A, j \rangle = \delta_{AB} \delta_{m_A m_B} \delta_{jk}. \quad \text{(B.17)}$$

C Two actions of the symmetric group

We introduced two Brauer algebras to label the orthogonal basis. One was responsible for flavour indices and the other was for colour indices. They have the same sub-algebra, i.e. the group algebra of the symmetric group $S_m \times S_n$, but they allow the sub-algebra to act

on the operator in different ways. The two different ways are correlated as we shall see in (C.3). Because the orthogonal basis should respect it, the Clebsh-Gordan coefficient should also satisfy a relation. We shall see that this is the case.

The symmetric group associated with colour indices act on the covariant operator as

$$
\sigma \left(O_{\Lambda, M_\Lambda, \Lambda_1, m_{\Lambda_1}, \tau} \right)^I_J \sigma^{-1} = \left(O_{\Lambda, M_\Lambda, \Lambda_1, m_{\Lambda_1}, \tau} \right)^{\sigma(I)}_{\sigma(J)}
$$
$$
= \sum_{\vec{a}, \vec{b}} C^{\vec{a}, \vec{b}}_{\Lambda, M_\Lambda, \Lambda_1, m_{\Lambda_1}, \tau} \left(O_{\vec{a}, \vec{b}} \right)^{\sigma(I)}_{\sigma(J)}.
\tag{C.1}
$$

On the other hand, the other symmetric group associated with flavour indices act on the covariant operator as

$$
D^{\Lambda_1}_{m_{\Lambda_1} m'_{\Lambda_1}}(\sigma) O_{\Lambda, M_\Lambda, \Lambda_1, m'_{\Lambda_1}, \tau} = \sum_{\vec{a}, \vec{b}} C^{\sigma(\vec{a}), \sigma(\vec{b})}_{\Lambda, M_\Lambda, \Lambda_1, m_{\Lambda_1}, \tau} \left(O_{\vec{a}, \vec{b}} \right)
$$
$$
= \sum_{\vec{a}, \vec{b}} C^{\vec{a}, \vec{b}}_{\Lambda, M_\Lambda, \Lambda_1, m_{\Lambda_1}, \tau} \left(O_{\sigma^{-1}(\vec{a}), \sigma^{-1}(\vec{b})} \right).
\tag{C.2}
$$

These two actions are not independent because of

$$
\left(O_{\vec{a}, \vec{b}} \right)^{\sigma(I)}_{\sigma(J)} = \left(O_{\sigma(\vec{a}), \sigma(\vec{b})} \right)^I_J.
\tag{C.3}
$$

This means the following equation to the covariant operator

$$
O_{\Lambda, M_\Lambda, \Lambda_1, m_{\Lambda_1}, \tau} = D^{\Lambda_1}_{m_{\Lambda_1} m'_{\Lambda_1}}(\sigma) \sigma \left(O_{\Lambda, M_\Lambda, \Lambda_1, m'_{\Lambda_1}, \tau} \right) \sigma^{-1}.
\tag{C.4}
$$

Substituting this in the gauge invariant operator (3.12), we obtain

$$
\mathcal{O}^{\gamma, A, ij, \tau_{\Lambda_1}}_{\Lambda, M_\Lambda, \Lambda_1, \tau} = C^{\tau_{\Lambda_1}, \Lambda_1, m_{\Lambda_1}}_{A, m_A, A, m'_A} \sum_b D^\gamma_{A, m_A i, m'_A j}(b^*) D^{\Lambda_1}_{m_{\Lambda_1} m'_{\Lambda_1}}(\sigma) tr_{m,n}(b \sigma O_{\Lambda, M_\Lambda, \Lambda_1, m'_{\Lambda_1}, \tau} \sigma^{-1})
$$
$$
= C^{\tau_{\Lambda_1}, \Lambda_1, m_{\Lambda_1}}_{A, m_A, A, m'_A} \sum_b D^\gamma_{A, m_A i, m'_A j}((\sigma b \sigma^{-1})^*) D^{\Lambda_1}_{m_{\Lambda_1} m'_{\Lambda_1}}(\sigma) tr_{m,n}(b O_{\Lambda, M_\Lambda, \Lambda_1, m'_{\Lambda_1}, \tau}).
$$

This σ-dependence can be shown to vanish as expected from the consistency of this operator. Using[11]

$$
(\sigma b \sigma^{-1})^* = \sigma b^* \sigma^{-1},
\tag{C.5}
$$

we can show that the σ-dependence disappears as

$$
C^{\tau_{\Lambda_1}, \Lambda_1, m_{\Lambda_1}}_{A, m_A, A, m'_A} \sum_b D^\gamma_{A, m_A i, m'_A j}((\sigma b \sigma^{-1})^*) D^{\Lambda_1}_{m_{\Lambda_1} m'_{\Lambda_1}}(\sigma)
$$
$$
= C^{\tau_{\Lambda_1}, \Lambda_1, m_{\Lambda_1}}_{A, m_A, A, m'_A} \sum_b D^A_{m_A m''_A}(\sigma) D^\gamma_{A, m''_A i, m'''_A j}(b^*) D^A_{m'''_A m'_A}(\sigma^{-1}) D^{\Lambda_1}_{m_{\Lambda_1} m'_{\Lambda_1}}(\sigma)
$$
$$
= C^{\tau_{\Lambda_1}, \Lambda_1, m''_{\Lambda_1}}_{A, m''_A, A, m'''_A} D^{\Lambda_1}_{m''_{\Lambda_1} m_{\Lambda_1}}(\sigma^{-1}) D^{\Lambda_1}_{m_{\Lambda_1} m'_{\Lambda_1}}(\sigma) \sum_b D^\gamma_{A, m''_A i, m'''_A j}(b^*)
$$

---

[11] This comes from the expression of b^* in (3.27) of [1]. with the assist of $\Sigma(\sigma b \sigma^{-1}) = \sigma \Sigma(b) \sigma^{-1}$, where $\Sigma(\sigma b \tau) = \sigma \Sigma(b) \tau$ is not true.

$$= C^{\tau_{\Lambda_1}, \Lambda_1, m'_{\Lambda_1}}_{A, m''_A, A, m'''_A} \sum_b D^{\gamma}_{A, m''_A i, m'''_A j}(b^*). \tag{C.6}$$

The final step comes from the following equation

$$\begin{aligned}
C^{\tau_{\Lambda_1}, \Lambda_1, m_{\Lambda_1}}_{A, m_A, A, m'_A} &D^A_{m_A m''_A}(\sigma) D^A_{m'''_A m'_A}(\sigma^{-1}) \\
&= C^{\tau_{\Lambda_1}, \Lambda_1, m_{\Lambda_1}}_{A, m_A, A, m'_A} D^A_{m''_A m_A}(\sigma^{-1}) D^A_{m'''_A m'_A}(\sigma^{-1}) \\
&= C^{\tau_{\Lambda_1}, \Lambda_1, m''_{\Lambda_1}}_{A, m''_A, A, m'''_A} D^{\Lambda_1}_{m''_{\Lambda_1} m_{\Lambda_1}}(\sigma^{-1}).
\end{aligned} \tag{C.7}$$

See appendix A in [13] for the derivation.

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